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On a Nonlinear Characterization Problem for Monosplines

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INTRODUCTION

In this paper we develop new methods for studying monosplines. In particular we develop a general theory for characterizing monosplines of least norm. Karlin announced results on the existence and characterization of monosplines of least norm in [11]. The present paper arose from our attempts to furnish proofs of these results and extend them. (We have since been informed by Karlin that he has given proofs in [23]). In another direction, we have extended these results to the problem of optimal quadrature formulas for analytic functions in [3]. The present paper complements Karlin's results by showing not only that there does exist a totally positive monospline of least norm that has simple knots but also that every monospline of least norm has this property. Our general approach is new and uses some tools developed by Braess [6, 7] and the concept of extended varisolvence [1].

In the first part of this paper we consider smooth monosplines. In the second part we treat polynomial monosplines by using smoothing techniques and the methods of the first part. Although it would have been possible to treat polynomial monosplines directly, we felt that dividing the paper in two parts was warranted because: 1) The methods used in both parts of the paper are the same, but they are much easier to understand in the first part, and 2) the results of the first part have intrinsic interest. Our treatment of polynomial monosplines does not include the uniform norm. Johnson [22] and Schumaker [2] have treated this case. In a future paper we plan to develop a general procedure for treating such problems.

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1. Extended Totally Positive Monosplines

In this part of the paper we will restrict our attention to an extended totally positive kernel $K(x, y)$ as defined in [10, 11]. We will assume that $K(x, y)$ is defined on $Y \times Y$, where Y is an open interval on the real line (possibly infinite) which contains [0, 1]. We will further assume that all derivatives occurring in Eqs. (1.1), (1.4) below are continuous.

The problem we deal with in this part of the paper is the following:

Problem. For any $1 \leqslant p < \infty$, let $\| \ \|$ be the $L_p[0, 1]$ norm and $F(x) = \int_0^1 K(x, y)\, dy$. For fixed positive integers N, M_0, M_1 we seek the $g(x)$ of the form

$$g(x) = \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} a_{ij} K^{(j)}(x, y_i) + \sum_{j=0}^{\overline{M}_0-1} a_j K^{(j)}(x, 0) + \sum_{j=0}^{\overline{M}_1-1} b_j K^{(j)}(x, 1), \quad (1.1)$$

with

$$\overline{M}_i \geqslant M_i \quad (i = 0, 1), \qquad \sum_{i=1}^{t} m_i + \overline{M}_0 + \overline{M}_1 \leqslant N + M_0 + M_1,$$

which minimizes

$$\| F + g \|. \quad (1.2)$$

More explicitly, $y_1 < y_2 < \cdots < y_t$, $y_i \neq 0, 1$, are free parameters all lying in Y and $\{a_{ij}, a_i, b_j\}$ are also free parameters with $a_{i,m_i-1} \neq 0$ $(i = 1,..., t)$, $a_{\overline{M}_0-1} \neq 0$ if $\overline{M}_0 > M_0$, and $b_{\overline{M}-1} \neq 0$ if $\overline{M}_1 > M_1$. $K^{(j)}(x, y) = (\partial^j/\partial y^j)\, K(x, y)$. Each y_i is called a free knot of g and $F + g$ is called a monospline.

Our strategy will be to show that for each finite closed interval $[c, d]$ such that $[0, 1] \subset (c, d) \subset Y$, there is a g of the form (1.1) which minimizes (1.2) when the y_i are restricted to lie in $[c, d]$. Further we will prove that any such minimizer has the property that its free knots are N in number and all lie in (0, 1). It will also be shown that all the coefficients associated with the free knots are strictly negative. Clearly, if the free knots are allowed to range over Y the same result holds.

We begin with a fixed finite closed interval $[c, d]$ in Y which properly contains [0, 1], we seek to minimize (1.2) when the free knots of g range over $[c, d]$.

Since the free knots vary over a compact set $[c, d]$, a slight modification of known results [18, Section 8.4] is that:

Lemma 1.1. *For $1 \leqslant p < \infty$ there exists an optimal g, with knots restricted to $[c, d]$, that is, a g which minimizes* (1.2).

DEFINITION 1.1. For some fixed p, $1 \leqslant p < \infty$, let $H(N, M_0, M_1, c, d)$ be all functions of the form (1.1) with knots $y_i \in [c, d]$. $h(x)$ is said to be in the gradient space of $g(x)$ if there are functions $G(u, x)$ and $h(x)$ satisfying the following conditions.

(a) $G(u, x) \in H(N, M_0, M_1, c, d)$, for $u \in [0, \delta]$, $\delta > 0$.

(b) $G(0, x) = g(x)$.

(c) $\partial G(u, x)/\partial u \,|_{u=0} = h(x)$.

(d) There are a $\delta > 0$ and a $k(x) \in L^1[0, 1]$ (depending on $G(u, x)$) such that for $u \in [0, \delta]$, $x \in [0, 1]$, $(\partial G/\partial u)(u, x)$ is separately continuous in u for almost all values of x, with $| F(x) + G(x, u)|^{p-1} |(\partial G/\partial u)(u, x)| \leqslant k(x)$.

For $1 \leqslant p < \infty$ the usual variational result, namely, Lemma 1.2, holds (see [15; 21, Theorem 46, p. 59]). Note that as an application of [2, Theorem 3] shows, $F + g$ has only a finite number of zeros, hence $p = 1$ causes no difficulties in the following lemma.

LEMMA 1.2. *If g is optimal*

$$\int_0^1 \operatorname{sgn}(F + g) \, | F + g |^{p-1} h \geqslant 0$$

for h in the gradient space of g.

Lemma 1.3 below describes the gradient space of g. For the statement of Lemma 1.3 we introduce the notation

$$\begin{aligned} \overline{\overline{M}}_i &= \overline{M}_i + 1 \qquad \text{if} \quad \overline{M}_i > M_i \qquad (i = 0, 1), \\ &= \overline{M}_i \qquad\quad\;\; \text{if} \quad \overline{M}_i = M_i \qquad (i = 0, 1). \end{aligned}$$

Also, m_c is the multiplicity of c as a free knot of g, m_d is the multiplicity of d as a free knot of g, $I = \{ y_i : y_i \text{ free knot of } g : y_i \neq 0, 1, c, d\}$, and p is the number of elements in I.

Further,

$$d(g) = \sum_{i=1}^{p} (m_i + 1) + \overline{\overline{M}}_0 + \overline{\overline{M}}_1 + m_c + m_d + l \tag{1.3}$$

where

$$l = N + M_0 + M_1 - \sum_{i=1}^{p} m_i - \overline{M}_0 - \overline{M}_1 - m_c - m_d .$$

$d(g)$ is called the degree of varisolvence of g.

For the statement of Lemma 1.3, we also need the following definition of the space $H(g)$.

Choose l points $0 < \bar{y}_1 < \bar{y}_2 < \cdots < \bar{y}_l < 1$ which are distinct from the free knots of g. Then consider the $d(g)$-dimensional subspace of $C[0, 1]$:

$$H(g) = \Big\{ \sum_{i=1}^{l} c_i K(x, \bar{y}_i) + \sum_{i \in I} \sum_{j=0}^{m_i} c_{ij} K^{(j)}(x, y_i)$$
$$+ \sum_{j=0}^{m_c - 1} p_j K^{(j)}(x, c) + \sum_{j=0}^{m_d - 1} q_j K^{(j)}(x, d) \tag{1.4}$$
$$+ \sum_{j=0}^{\overline{\overline{M}}_0 - 1} d_j K^{(j)}(x, 0) + \sum_{j=0}^{\overline{\overline{M}}_1 - 1} e_j K^{(j)}(x, 1);$$
$$p_j, q_j, c_i, c_{ij}, d_j, e_j \text{ free real parameters} \Big\}.$$

Clearly $H(g)$ has a basis which forms a Markoff system [9, p. 76].

LEMMA 1.3. (a) *The gradient space of g of the form* (1.1) *contains* $H(g) \oplus \text{cone}(g)$, *where $H(g)$ is the $d(g)$ dimensional linear space described in* (1.3) *and* (1.4), *and* cone(g) *is a cone described below.*

(b) *If g has a free knot at c {resp. d} of multiplicity m_c {resp. m_d} then* cone(g) *includes the functions*

$$p_{m_c} K^{(m_c)}(x, c) \qquad \{\text{resp. } p_{m_c} K^{(m_d)}(x, d)\} \tag{1.5}$$

with the restriction

$$\operatorname{sgn} p_{m_c} = \operatorname{sgn} a_{1, m_c - 1}, \qquad \text{where} \quad y_1 = c \tag{1.6}$$

{resp. $\operatorname{sgn} p_{m_d} = -\operatorname{sgn} a_{t, m_d - 1}$, *where* $y_t = d$*}.*

(c) *If g has a free knot at y_q, $0 < y_q < 1$ with multiplicity $m_q > 1$, then* cone(g) *contains the functions*

$$c_{q, m_q + 1} K^{(m_q + 1)}(x, y_q) \tag{1.7}$$

with the restriction

$$\operatorname{sgn} c_{q, m_q + 1} = \operatorname{sgn} a_{q, m_q - 1}. \tag{1.8}$$

Proof. Parts (a) and (b) follow immediately by letting the parameters $\{a_{ij}, a_j, b_j, y_j\}$ in (1.1) vary with u in the case $\overline{\overline{M}}_i = M_i$ $(i = 0, 1)$. We will return to the other case after proving part (c).

Part (c) is proved in [7]. However, because of its importance for our considerations we give another proof. Our proof has the further advantage that it generalizes to splines (see Lemma 2.4).

Clearly it will be sufficient to prove the result for

$$g(x) = \sum_{j=0}^{m-1} a_j K^{(j)}(x, y_q). \tag{1.9}$$

We begin by letting $p(y)$ be the unique polynomial of degree at most $m - 1$, with real coefficients such that

$$g(x) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dy^{m-1}} (p(y) K(x, y)) \Big|_{y=y_q}.$$

If $K(x, y)$ is not analytic in y, we can approximate it by functions $\{K_\epsilon(x, y)\}$ that are analytic in y and such that

$$\begin{aligned} &\lim_{\epsilon\to 0} K_\epsilon^{(j)}(x, y) \to K^{(j)}(x, y) \text{ uniformly on} \\ &[0, 1] \times [0, 1] \quad \text{for } j = 0, 1,..., m + 1. \end{aligned} \tag{1.10}$$

(See the second part of this paper.)

Let Γ be a positively oriented circle with y_q as center. We consider sufficiently small $u > 0$, so that $y_q \pm u^{1/2}$ lie strictly inside of Γ. We then define for $u > 0$:

$$\begin{aligned} G_\epsilon(x, u) &= \frac{1}{2\pi i} \int_\Gamma \frac{p(z) K_\epsilon(x, z)\, dz}{(z - y_q)^{m-2}((z - y_q)^2 - u)} \\ &= \sum_{j=0}^{m-3} d_{0j}(u) K_\epsilon^{(j)}(x, y_q) + d_1(u) K_\epsilon(x, y_q + u^{1/2}) \\ &\quad + d_2(u) K_\epsilon(x, y_q - u^{1/2}) \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} \frac{\partial G_\epsilon(x, u)}{\partial u} &= \frac{1}{2\pi i} \int_\Gamma \frac{p(z) K_\epsilon(x, z)\, dz}{(z - y_q)^{m-2}((z - y_q)^2 - u)^2} \\ &= \sum_{j=0}^{m-3} l_{0j}(u) K_\epsilon^{(j)}(x, y_q) \\ &\quad + \sum_{j=0}^{1} [l_{1j}(u) K_\epsilon^{(j)}(x, y_q + u^{1/2}) + l_{2j}(u) K_\epsilon^{(j)}(x, y_q - u^{1/2})]. \end{aligned} \tag{1.12}$$

This follows by a partial fraction decomposition, and Cauchy's Integral Formula.

Since the right-hand sides of (1.11) and (1.12) do not depend on the $K_\epsilon(x, y)$ being analytic, we can replace the $K_\epsilon(x, y)$ by $K(x, y)$ and thus define two new functions $G(x, u)$, $H(x, u)$, respectively (see (1.10)). Because these expressions are really divided differences, we have by the linearity of divided differences and the generalized mean value theorem for divided differences (see [16, Chap. III, Problem 164, Chap. V, Problem 97]):

$$aG(x, u) + bG_\epsilon(x, u) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dy^{m-1}} (p(y)(aK(x, y) + bK_\epsilon(x, y)) \Big|_{y=\tilde{y}},$$
$$y_q - u^{1/2} < \tilde{y} < y_q + u^{1/2} \quad (1.13)$$

$$aH(x, u) + b \frac{\partial G_\epsilon}{\partial u}(x, u) = \frac{1}{(m+1)!} \frac{d^{m+1}}{dy^{m+1}} (p(y)(aK(x, y) + bK_\epsilon(x, y)) \Big|_{y=\tilde{\tilde{y}}},$$
$$y_q - u^{1/2} < \tilde{\tilde{y}} < y_q + u^{1/2} \quad (1.14)$$

($\tilde{y}$ and $\tilde{\tilde{y}}$ depend on a and b, $\tilde{y} = \tilde{\tilde{y}} = y_q$ when $u = 0$).

For $a = 1$, $b = -1$ in (1.13), (1.14) we find using (1.10) that for fixed x, $G_\epsilon(x, u)$ uniformly approaches $G(x, u)$ and $(\partial G_\epsilon/\partial u)(x, u)$ uniformly approaches $H(x, u)$. Thus we can assert that $(\partial G/\partial u)(x, u) = H(x, u)$ and $(\partial G/\partial u)(x, u)$ is continuous in u for $x \in [0, 1]$, $u \in [0, \delta]$.

Further, by setting $a = 1$, $b = 0$ in (1.13), (1.14), we find that $G(x, u)$ and $(\partial G/\partial u)(x, u)$ are uniformly bounded, and that

$$\begin{aligned} &\lim_{u \to 0} G(x, u) = g(x), \\ &\lim_{u \to 0} \frac{\partial G(x, u)}{\partial u} = \frac{1}{(m+1)!} \frac{d^{m+1}}{dy^{m+1}} (p(y)\, K(x, y)) \Big|_{y=y_q}. \end{aligned} \quad (1.15)$$

Hence (d) of Definition 1.1 is verified, for this $G(x, u)$. Thus we have shown the function on the right side of (1.15) is in the gradient space of $g(x)$. To get the complete result, we can easily extend the above analysis of (1.11) to

$$\tilde{G}_\epsilon(x, u) = \frac{1}{2\pi i} \int_\Gamma \frac{p(z) + up_1(z)}{(z-y)^{m-2}((z-y)^2 - w)} K_\epsilon(x, z)\, dz \quad (1.16)$$

where $w = \lambda_1 u$, $y = y_q + \lambda_2 u$, and $p_1(z)$ is an arbitrary polynomial with real coefficients of degree $m - 1$; λ_1, λ_2 constants; $\lambda_1 > 0$. Note

$$\begin{aligned} \tilde{G}_\epsilon(x, 0) &= g(x), \\ \frac{\partial \tilde{G}_\epsilon(x, u)}{\partial u} \Big|_{u=0} &= \lambda_1 \frac{1}{(m+1)!} \frac{d^{m+1}}{dy^{m+1}} (p(y)\, K_\epsilon(x, y)) \Big|_{y=y_q} \\ &\quad + \lambda_2 \frac{1}{(m-1)!} \frac{d^m}{dy^m} (p(y)\, K_\epsilon(x, y)) \Big|_{y=y_q} \\ &\quad + \frac{1}{(m-1)!} \frac{d^{m-1}}{dy^{m-1}} (p_1(y)\, K_\epsilon(x, y)) \Big|_{y=y_q}. \end{aligned} \quad (1.17)$$

This completes the proof of part (c).

We now return to the proof of part (a) when either $\bar{\bar{M}}_2 > \bar{M}_0$ or $\bar{\bar{M}}_1 > \bar{M}_1$. For example, if $\bar{\bar{M}}_0 > \bar{M}_0 = m$, then as in part (c) it suffices to prove the result for

$$g(x) = \sum_{j=0}^{m-1} a_j K^{(j)}(x, 0).$$

We follow the proof of part (c) but in (1.11) we replace $G_\epsilon(x, u)$ by

$$G_\epsilon(x, u) = \frac{1}{2\pi i} \int_\Gamma \frac{p(z)\, K_\epsilon(x, z)\, dz}{z^{m-1}(z - u)},$$

where u varies in both directions from zero. Proceeding as in part (c), it follows that the function $\partial G(x, u)/\partial u$ defined in (1.15) is in the unrestricted gradient space of $g(x)$. In (1.16), if one makes the obvious modification it is clear then that the function

$$\sum_{j=0}^{\bar{\bar{M}}_0 - 1} d_j K^{(j)}(x, 0),$$

where the $\{d_j\}$ are an arbitrary set of real numbers, is in the gradient space of $g(x)$ (see (1.4)). This completes the proof. ∎

Definition 1.2. We say that $f(x)$ a continuous function defined for $0 \leqslant x \leqslant 1$ has k sign changes or $S^-(f) = k$ if there is a maximal set of $k + 1$ points x_i, with $0 \leqslant x_0 < x_1 < \cdots < x_k \leqslant 1$ such that $f(x_i) f(x_{i+1}) < 0$, $i = 0, \ldots, k$.

Note. In the last half of this paper we extend this definition to piecewise continuous functions $f(x)$, with the proviso that none of the x_i's used in the definition are points of discontinuity of $f(x)$.

Lemma 1.4. *Let H be a subspace of $C[0, 1]$ of dimension $n \geqslant k + 1$, for which there is a basis which forms a Markoff system. If $f \in C[0, 1]$ has only a finite number of zeros with $S^-(f) = k$, then there is an $h_1 \in H$, so $h_1(x) = 0 \Rightarrow f(x) = 0$ and for which $h_1 f \leqslant 0$ and $\int_0^1 h_1 f < 0$.*

Proof. We assume that the dimension of H is $k + 1$, for if not we consider a $(k + 1)$-dimensional subspace which has a Markoff basis. If x_i, $i = 0, \ldots, k$, is a maximal set of points for the sign changes of f, choose h_1 to vanish at the k points t_i, $i = 0, \ldots, k - 1$, where $t_i = \text{lub}\{x \mid x_i < x < x_{i+1},\ \operatorname{sgn} f(x) = \operatorname{sgn} f(x_i)\}$. Then $h_1(x)$ not identically zero is uniquely determined by further requiring $h_1(x_0) = -f(x_0) \neq 0$.

It now follows, from the previous lemmas, that

LEMMA 1.5. *For* $1 \leqslant p < \infty$, g optimal implies $S^{-}(F+g) \geqslant d(g)$.

A device introduced by Braess [6] will be useful. Let

$$h_2(x) = \sum_{i=1}^{t} \sum_{j=0}^{m_i} h_{ij} K^{(j)}(x, y_i), \tag{1.18}$$

where

$$M = \sum_{i=1}^{t} m_i + 1 \quad h_{i,m_i} \neq 0 \qquad (i = 1,..., t), \text{ and } c \leqslant y_1 < y_2 < \cdots < y_t \leqslant d.$$

DEFINITION 1.3. The generalized sign vector V of $h_2(x)$ is the M tuple of ± 1 defined as

$$V(h_2) = ((-1)^{m_1} s_1, (-1)^{m_1-1} s_1,..., (-1) s_1, s_1, (-1)^{m_2} s_2,...,$$
$$s_2,..., s_{t-1}, (-1)^{m_t} s_t,..., s_t) = (V_0, V_1,..., V_{M-1}), \tag{1.19}$$

where $s_i = \operatorname{sgn} h_{i,m_i}$ $(i = 1,..., t)$. We say h_2 has k generalized sign changes or $V^{-}(h_2) = k$ if there are a maximal number of $k+1$ coordinates of $V(h_2)$, $V_{i_0}, V_{i_1}, V_{i_2},..., V_{i_k}$ where $i_j < i_{j+1}$ $(j = 0,..., k-1)$ such that $V_{i_j} V_{i_{j+1}} = -1$ $(j = 0,..., k-1)$. In this case we also say that $V(h_2)$ has k sign changes.

The following result will be used.

LEMMA 1.6 (Braess [6]). *Let* $h_2(x)$ *be as in* (1.18). *Then*:

(a) $S^{-}(h_2) \leqslant V^{-}(h_2) \leqslant M - 1$.

(b) *If* $S^{-}(h_2) = V^{-}(h_2) = k$, *then* $\operatorname{sgn} h_2(x_j) = V_{i_j}$, $j = 0,..., k$, *where the* x_j *are a maximal set of coordinates for the sign changes of* $h_2(x)$ *and the* V_{i_j} *are a maximal number of coordinates for the sign changes in* $V(h_2)$.

We also need a generalization:

DEFINITION 1.4. Let

$$F + g_1 = \int_0^1 K(x, y)\, dy + \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} a_{ij} K^{(j)}(x, y_i) \tag{1.20}$$

where $c \leqslant y_1 < y_2 < \cdots < y_s \leqslant d$; $y_{s'} \leqslant 0 < y_{s'+1}$, and $y_{s''-1} < 1 \leqslant y_{s''}$, $s' < s''$, and $\alpha_i = \operatorname{sgn} a_{i,m_i-1} \neq 0$ $(i = 1,..., s)$. If $D(F+g_1) = \sum_{i=1}^{s} (m_i + 1)$ and $C(F+g_1)$ is the number of y_i which are not in $(0, 1)$, then we define the generalized sign vector $V(F+g_1)$ (which has $D(F+g_1) - C(F+g_1) + 1$ components consisting of ± 1) by

$$
\begin{aligned}
V(F+g_1) &= (\overbrace{(-1)^{m_1-1}\alpha_1,\ldots,(-1)\alpha_1,\alpha_1}^{m_1},(-1)^{m_2-1}\alpha_2,\ldots,\alpha_{s'},\\
&\quad \boxed{+1},\overbrace{(-1)^{m_{s'+1}-1}\alpha_{s'+1},\ldots,(-1)\alpha_{s'+1},\alpha_{s'+1}}^{m_{s'+1}},\boxed{+1},\\
&\quad \ldots,\boxed{+1},\underbrace{(-1)^{m_{s''-1}}\quad\alpha_{s''-1},\ldots,(-1)\alpha_{s''-1},\alpha_{s''-1}}_{m_{s''-1}},\boxed{+1},(-1)^{m_{s''}-1}\alpha_{s''},\\
&\quad \ldots,(-1)\alpha_{s''},\alpha_{s''},(-1)^{m_{s''+1}-1}\alpha_{s''+1},\ldots,\alpha_s)\\
&= (V_0,\ldots,V_{D(F+g_1)-C(F+g_1)}). \qquad (1.21)
\end{aligned}
$$

The $\boxed{+1}$ are inserted immediately before each $(-1)^{m_j-1}\alpha_j$, $j = s'+1,\ldots, s''-1$, and also immediately after $\alpha_{s''-1}$. (This takes care of the possibilities that all knots are greater than 0 and/or less than 1.)

We say $F + g_1$ has k generalized sign changes or $V^-(F+g_1) = k$ if there are a maximal number of $k+1$ coordinates of $V(F+g_1)$, $V_{i_0}, V_{i_1},\ldots, V_{i_k}$ where $i_j < i_{j+1}$ $(j = 0,\ldots, k-1)$ such that $V_{i_j}V_{i_{j+1}} = -1$ $(j = 0,\ldots, k-1)$. In this case we also say that $V(F+g_1)$ has k sign changes.

LEMMA 1.7. *With $F+g_1$, $V(F+g_1)$, $D(F+g_1)$, and $C(F+g_1)$ as in Definition* 1.4, *and where $E(F+g_1)$ is the number of knots of even multiplicity in* (0, 1):

(a) $S^-(F+g_1) \leqslant V^-(F+g_1) \leqslant D(F+g_1) - C(F+g_1) - E(F+g_1)$.

(b) *If $S^-(F+g_1) = V^-(F+g_1) = k$, then* $\operatorname{sgn}(F+g_1)(x_j) = V_{i_j}$ $(j = 0,\ldots, k)$, *where the x_j are a maximal set of coordinates for the sign changes of $F+g_1$, and the V_{i_j} are a maximal number of coordinates for the sign changes in $V(F+g_1)$.*

Proof. We can obviously approximate $F(x)$ uniformly by Riemann sums:

$$\left\{F_n(x) = (1/n)\sum_{i=0}^{n-1} K(x, y_i^{(n)}),\ i/n < y_i^{(n)} < (i+1)/n\right\}, \qquad (1.22)$$

where none of the $y_i^{(n)}$ in (1.22) equals a y_i in (1.20).

For sufficiently large n, we have by Definitions 1.3 and 1.4 and Lemma 1.6

$$S^-(F+g_1) \leqslant S^-(F_n+g_1) \leqslant V^-(F_n+g_1) = V^-(F+g_1), \qquad (1.23)$$

which establishes the first part of (a).

To establish the second part we first note that $V(F + g_1)$ has $D(F + g_1) - C(F + g_1) + 1$ components. Corresponding to a knot $y_q \in (0, 1)$, the components of $V(F + g_1)$ look like:

$$(\boxed{+1}, (-1)^{m_q-1} \alpha_q ,..., (-1) \alpha_q , \alpha_q , \boxed{+1}). \tag{1.24}$$

If m_q is even, $(-1)^{m_q-1} \alpha_q \neq \alpha_q$, hence the vector (1.24) has m_q sign changes which is one less than is possible with $m_q + 2$ components. In a similar fashion at each even knot in (0, 1) the vector $V(F + g_1)$ loses one possible sign change. This establishes (a). (b) follows from (1.23) and Lemma 1.6.

LEMMA 1.8. *If g of the form* (1.1) *with knots restricted to lie in* $[c, d]$ *is optimal, then*

(a) *There are no free knots of $F + g$ in $(c, 0] \cup [1, d)$.*

(b) $N = \sum_{i=1}^{t} m_i$.

(c) *All free knots in* (0, 1) *are of odd multiplicity.*

(d) $S^-(F + g) = d(g) = D(F + g) - C(F + g)$.

Proof. From Lemmas 1.7 and 1.5, a necessary condition for g to be optimal is

$$D(F + g) - C(F + g) - E(F + g) \geqslant d(g). \tag{1.25}$$

We will show that if (a), (b), and (c) do not hold then (1.25) is violated.

Interpreting $D(F + g)$ and $C(F + g)$ of Definition (1.4) and Eq. (1.20) in terms of Eqs. (1.1) and (1.3) we have, letting $\Phi(g)$ be the number of free knots in $(c, 0) \cup (1, d)$,

$$\begin{aligned} D(F + g) - C(F + g) &= \sum_{i=1}^{p} (m_i + 1) - \Phi(g) + \overline{M}_0 + \overline{M}_1 + m_c + m_d \\ &\leqslant \sum_{i=1}^{p} (m_i + 1) + \overline{\overline{M}}_0 + \overline{\overline{M}}_1 + m_c + m_d + l = d(g). \end{aligned} \tag{1.26}$$

Note from Definition (1.2), (1.25), and (1.26) it follows that

(i) $\Phi(g) = 0$ and $\overline{\overline{M}}_i = \overline{M}_i$ $(i = 0, 1)$, which implies conclusion (a);

(ii) $l = 0$, which together with (i) yields conclusion (b).

Equations (1.25) and (1.26) yield immediately also that $E(F + g) = 0$, hence (c) holds. Finally (d) follows from Lemma 5, (1.25), and (1.26). ∎

LEMMA 1.9. *If g of the form* (1.1) *with knots restricted to* $[c, d]$ *is optimal then*

(a) *g has no free knots at c or d,*

(b) *all the free knots of g have multiplicity one,*

(c) *at a free knot y_i of multiplicity one,* $\operatorname{sgn} a_{i0} = -1$.

Proof. We first prove (a). For ease of exposition, assume g has a free knot at c of multiplicity m_c, but no free knot at d. Let $\tilde{H}(g)$ be the $d(g) + 1$ subspace, consisting of all functions of the form (1.4) and (1.5) without the restriction (1.6). From Lemma 1.8, $S^-(F + g) = d(g)$, hence we may apply Lemma 1.4 to find a function $h \in \tilde{H}(g)$ such that

$$\int_0^1 \operatorname{sgn}(F + g) \mid F + g \mid^{p-1} h < 0, \tag{1.27}$$

and points $x_i \in (0, 1)$, corresponding to the sign changes of $F + g$, such that

$$h(x_i)(F + g)(x_i) < 0, \qquad i = 0,..., d(g). \tag{1.28}$$

We now show that the restriction (1.6) holds. From (1.28), $S^-(h) \geqslant d(g)$, but since the dimension of $\tilde{H}(g)$ is $d(g) + 1$, it follows from Lemma 1.6, that $S^-(h) = d(g)$, and that

$$\operatorname{sgn} h(x_i) = V_i(h), \qquad i = 0,..., d(g). \tag{1.29}$$

Similarly, since Lemma 1.8(d) asserts that $S^-(F + g) = d(g) = D(F + g) - C(F + g)$, Lemma 1.7 implies

$$\operatorname{sgn}(F + g)(x_i) = V_i(F + g), \qquad i = 0,..., d(g). \tag{1.30}$$

Setting $i = 0$ in (1.29) and (1.30), and using Definitions 1.3 and 1.4 of the generalized sign vectors we find

$$\begin{aligned}(-1)^{m_c} \operatorname{sgn} p_{m_c} &= V_0(h) = \operatorname{sgn} h(x_0) = -\operatorname{sgn}(F + g)(x_0) \\ &= -V_0(F + g) = (-1)(-1)^{m_c - 1} \operatorname{sgn} a_{1,m_c-1} .\end{aligned} \tag{1.31}$$

This shows that (1.6) holds, hence $h(x)$ is in the gradient space of g, thus by Lemma 1.2, g is not optimal. The possibility of a free knot at (d) is ruled out using the same methods. Hence the proof of (a) is complete.

Part (b) is proved in a similar fashion. Let $g(x)$ be optimal, with at least one free knot of odd multiplicity three or greater. Consider the free knot $y_{t'}$, which is the largest free knot of odd multiplicity greater than one—that is, $m_{t'} \geqslant 3$, and $1 > y_i > y_{t'}$ implies $m_i = 1$.

We now consider the $d(g) + 1$ subspace $\tilde{H}(g)$ of all functions of the form (1.4) and (1.7), neglecting the restriction (1.8). Proceeding as above, we find a function $h(x) \in \tilde{H}(g)$ satisfying (1.27), (1.28), and (1.29), and as before (1.30) is satisfied. We now show that this $h(x)$ satisfies (1.8), which will complete the proof of part (b).

From (1.18), (1.29), and (1.30) it follows that all terms in $V(h)$ and $V(F + g)$ alternate in sign, and both have the same number of components.

We pair corresponding terms in $V(F + g)$ [with g written in the form (1.1)] with terms in $V(h)$ [with h written in the form (1.4) and (1.7)]. Starting from the right end in each vector, we pair

$$((-1)^{M_1-1} \operatorname{sgn} b_{M_1-1}, ..., \operatorname{sgn} b_{M_1-1}) \text{ in } V(F+g),$$

with (1.32)

$$((-1)^{M_1-1} \operatorname{sgn} e_{M_1-1}, ..., \operatorname{sgn} e_{M_1-1}) \text{ in } V(h).$$

For $y_i > y_{t'}$, we pair

$$(\operatorname{sgn}(a_{i0}), \boxed{+1}) \text{ in } V(F+g) \quad \text{with} \quad ((-1) \operatorname{sgn} c_{i1}, \operatorname{sgn} c_{i1}) \text{ in } V(h). \tag{1.33}$$

Finally, we pair

$$(\operatorname{sgn}(a_{t',m_{t'}-1}), \boxed{+1}) \text{ in } V(F+g) \quad \text{with} \quad ((-1) \operatorname{sgn} c_{t',m_{t'}+1}, \operatorname{sgn} c_{t',m_{t'}+1}) \text{ in } V(h). \tag{1.34}$$

From (1.34) it follows from (1.28)-(1.30), that there is a q such that

$$\operatorname{sgn}(a_{t',m_{t'}-1}) = V_q(F+g) = F + g(x_q) = -h(x_q) = -V_q(h) = \operatorname{sgn} c_{t',m_{t'}+1}. \tag{1.35}$$

Thus $h(x)$ satisfies (1.8) and this completes the proof of (b). (c) follows, since as mentioned above all terms in $V(F + g)$ alternate in sign. ∎

We now allow the parameters for g to range over an open set and summarize our results in the following theorem.

THEOREM 1.1. *For* $1 \leqslant p < \infty$ *there exists a best approximation* $g(x)$ *to* $F(x) = \int_0^1 K(x, y)\, dy$ *when the parameters for* $g(x)$ *are allowed to vary over an open set* Y *containing* $[0, 1]$. *Further, each best approximation is of the form*

$$g(x) = \sum_{j=0}^{M_0-1} a_j K^{(j)}(x, 0) + \sum_{i=1}^{N} a_{i0} K(x, y_i) + \sum_{j=0}^{M_1-1} b_j K^{(j)}(x, 1) \tag{1.36}$$

where

$$a_{M_0-1} < 0, \qquad (-1)^{M_1-1} b_{M_1-1} < 0, \qquad a_{i0} < 0 \, (i = 1,..., N),$$
$$0 < y_1 < y_2 < \cdots < y_N < 1. \tag{1.37}$$

Proof. Take any minimizing sequence $\{g_N(x)\}$, with knots restricted to lie in Y, i.e. $\lim_{N\to\infty} \| F + g_N \|$ approaches the infimum. By our previous discussion we know that for each N, we can find a $\tilde{g}_N$ with knots restricted to lie in $[0, 1]$ such that $\| F + \tilde{g}_N \| \leqslant \| F + g_N \|$. Thus from our lemmas an optimal approximation exists and any optimal approximation $g(x)$ is of the form (1.36), (1.37). ∎

It is quite clear that these results can be extended to include any finite positive measure, and the norm could be taken over any finite closed interval.

We have not treated the case of the uniform norm in this paper, but it should be apparent to the reader that with slight modification our analysis applies to this case; e.g., Lemma 5 for $p = \infty$ is proved in [18, p. 10].

2. Polynomial Monosplines

Using the previous results on extended totally positive monosplines and the technique of smoothing, we are also able to treat polynomial monosplines.

We will restrict ourselves to the following problem but our methods extend to more general problems; see [13].

Problem. For any $1 \leqslant p < \infty$, let $\| \ \|$ be the $L_p[0, 1]$ norm and $F(x) = \int_0^1 (x - y)_+^{n-1} \, dy$. For fixed integers N and n we seek the $g(x)$ of the form

$$g(x) = \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} r_{ij}(x - y_i)_+^{n-1-j}, \tag{2.1}$$

which minimizes

$$\| F + g \|. \tag{2.2}$$

Here $\sum_{i=1}^t m_i \leqslant N$, $\{a_i, r_{ij}, y_i\}$ are free real parameters, $r_{i,m_i-1} \neq 0$, $m_i \leqslant n$, $i = 1,..., t$, $0 < y_1 < y_2 < \cdots < y_t < 1$.

Our first result is

Lemma 2.1. *For* $1 \leqslant p < \infty$, *there exists an optimal* g, *i.e., a* g *of the form* (2.1) *that minimizes* (2.2). *Further, for* g *to be optimal it is necessary that* g *be continuous, hence we may assume* $m_i \leqslant n - 1$ *in* (2.1).

Proof. The existence of a best approximation is proved, for example, in [4, 18, 20]. That an optimal g in the L_2 norm is continuous is proved in [17, Theorem 5]. This proof extends to the L_p norm $1 \leqslant p < \infty$. ∎

Let $H(N, n)$ be all functions of the form (2.1) with $m_i \leqslant n - 1$. With $H(N, M_0, M_1, c, d)$ of Definition 1.1 replaced by $H(N, n)$, the definition carries over to define the gradient space of g.

Since $F + g$ is a nonzero polynomial between knots, it has only a finite number of zeros. Hence we obtain as previously,

LEMMA 2.2. *If g is optimal, $1 \leqslant p < \infty$ then*

$$\int_0^1 \operatorname{sgn}(F + g) \mid F + g \mid^{p-1} h \geqslant 0$$

for h in the gradient space of g.

Before proceeding, we quote a result on smoothing (see [10, pp. 512–513; 5]).

LEMMA 2.3. *Let*

$$L^{(j)}(x, y) = (x - y)_+^{n-1-j}/n - 1 - j!, \qquad j = 0,\dots, n - 1,$$

$$L(x, y) = L^{(0)}(x, y),$$

$$G_\epsilon(z) = (1/(2\pi)^{1/2}\,\epsilon) \exp(-z^2/2\epsilon), \qquad \epsilon > 0,$$

$$K_\epsilon^{(j)}(x, y) = \int_{-\infty}^{\infty} G_\epsilon(x - \xi)\, L^{(j)}(\xi, y)\, d\xi, \qquad \epsilon > 0, \quad j = 0,\dots, n - 1,$$

$$K_\epsilon(x, y) = K_\epsilon^{(0)}(x, y).$$

Then differentiation under the integral sign is permissible, that is,

$$(\partial^j/\partial y^j)\, K_\epsilon(x, y) = K_\epsilon^{(j)}(x, y), \qquad j = 0,\dots, n - 1.$$

Further, for any L_p norm $1 \leqslant p < \infty$:

$$\lim_{\epsilon \to 0} \| K_\epsilon^{(j)}(x, y) - L^{(j)}(x, y)\| \to 0, \qquad j = 0,\dots, n - 1. \tag{2.3}$$

$\lim_{\epsilon \to 0} K_\epsilon^{(j)}(x, y) \to L^{(j)}(x, y)$ (a) *uniformly on all compact subsets of $\mathbb{R}^2$ for $j \leqslant n - 2$,*
(b) *uniformly on all compact subsets of $\mathbb{R}^2$ not intersecting the diagonal $x = y$ for $j = n - 1$.* (2.4)

The kernel $K_\epsilon(x, y)$ is analytic in x and y and extended totally positive, for $\epsilon > 0$. (2.5)

$| K_\epsilon^{(n-1)}(x, y)| \leqslant 1$, for $\epsilon > 0$. $| K_\epsilon^{(j)}(x, y)|$ is bounded for (x, y) belonging to a compact set of R^2, $j = 0,\dots, n - 2$, $\epsilon > 0$. (2.6)

LEMMA 2.4. (a) *The gradient space of g of the form* (2.1) *with $m_i \leqslant n-1$ contains $H(g) \oplus \operatorname{cone}(g)$, where $H(g)$ is the $d(g)$-dimensional linear space of all functions $h(x)$ with*

$$d(g) = n + \sum_{i=1}^{t} (m_i + 1) + \left(N - \sum_{i=1}^{t} m_i\right), \tag{2.7}$$

$$h(x) = \sum_{i=0}^{n-1} b_i x^i + \sum_{i=1}^{t} \sum_{j=0}^{m_i} s_{ij}(x - y_i)_+^{n-1-j} + \sum_{i=1}^{l} c_i(x - \bar{y}_i)_+^{n-1}, \tag{2.8}$$

where $l = N - \sum_{i=1}^{t} m_i$, and $0 < \bar{y}_1 < \bar{y}_2 \cdots < \bar{y}_l < 1$ are distinct from the free knots of g.

(b) *If g has a free knot at y_q, $0 < y_q < 1$, with multiplicity m_q, m_q, $1 < m_q \leqslant n-2$, then* cone(g) *contains the functions*

$$s_{q,m_q+1}(x - y_q)_+^{(n-1)-(m_q+1)} \tag{2.9}$$

with the restriction

$$\operatorname{sgn} s_{q,m_q+1} = \operatorname{sgn} r_{q,m_q-1} . \tag{2.10}$$

Proof. The major difficulty occurs in part (b) when $m_q = n-2$, we will restrict ourselves to this case. We proceed as in the proof of part (c) of Lemma 1.3, and borrow some of the notation used there.

Analogous to (1.11) we discuss the integral

$$G_\epsilon(x, u) = \frac{1}{2\pi i} \int_\Gamma \frac{p(z)\, K_\epsilon(x, z)\, dz}{(z - y_q)^{n-4}\, ((z - y_q)^2 - u)},$$

where $K_\epsilon(x, z)$ is defined in Lemma 2.3.

Replacing $K_\epsilon(x, y)$ in the right sides of (1.11) and (1.12) by $L(x, y)$, we define two functions $G(x, u)$ and $H(x, u)$ and proceed exactly as in the proof of part (c) of Lemma 1.3.

We now replace (1.13) by

$$\begin{aligned} &aG(x, u) + bG_\epsilon(x, u) = \frac{1}{(n-3)!} \frac{d^{n-3}}{dy^{n-3}} (p(y)(aL(x, y) + bK_\epsilon(x, y)))|_{y=\tilde{y}}, \\ &y_q - u^{1/2} < \tilde{y} < y_q + u^{1/2}, \qquad \text{and } \tilde{y} = y_q \qquad \text{if } u = 0. \end{aligned} \tag{2.11}$$

We replace (1.14) for fixed $x \neq y_q$ if $u^{1/2} < |x - y_q|$, $u > 0$ (and for $x \in [0, 1]$, $u \in [0, \delta]$ if $a = 0$) by

$$\begin{aligned} &aH(x, u) + b\frac{\partial G_\epsilon}{\partial u}(x, u) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dy^{n-1}} (p(y)(aL(x, y) + bK_\epsilon(x, y)))|_{y=\tilde{\tilde{y}}}, \\ &y_q - u^{1/2} < \tilde{\tilde{y}} < y_q + u^{1/2}, \qquad \text{and} \qquad \tilde{\tilde{y}} = y_q \qquad \text{if } u = 0. \end{aligned} \tag{2.12}$$

While if $u^{1/2} \geqslant | x - y_q |$, we replace (1.14) by

$$aH(x, u) + b \frac{\partial G_\epsilon}{\partial u}(x, u) = \sum_{j=0}^{n-5} l_{0j}(u)(aL^{(j)}(x, y_q) + bK_\epsilon^{(j)}(x, y_q))$$

$$+ \sum_{j=0}^{1} [l_{1j}^{(j)}(aL^{(j)}(x, y_q + u^{1/2}) + bK_\epsilon^{(j)}(x, y_q + u^{1/2})) \tag{2.13}$$

$$+ l_{2j}(u)(aL^{(j)}(x, y_q - u^{1/2}) + bK_\epsilon^{(j)}(x, y_q - u^{1/2})].$$

Setting $a = 0$, $b = 1$ in (2.11) and (2.12) and using (2.6) we find that $G_\epsilon(x, u)$, and $(\partial G_\epsilon/\partial u)(x, u)$ are uniformly bounded for $x \in [0, 1]$, $u \in [0, \delta]$.

For fixed $x \neq y_q$, setting $a = 1$, $b = -1$ in these equations, we note that since the coefficients $l_{0j}(u)$, $l_{1j}(u)$, and $l_{2j}(u)$ of (2.13) are bounded for $u^{1/2} \geqslant | x - y_q |$, that $G_\epsilon(x, u)$ uniformly approaches $G(x, u)$ and $(\partial G_\epsilon/\partial u)(x, u)$ uniformly approaches $H(x, u)$. Hence we obtain the result that $(\partial G/\partial u)(x, u)$ exists and is continuous in u for fixed $x \neq y_q$; moreover, it and $G(x, u)$ are uniformly bounded. Thus (d) of Definition (1.1) is satisfied for this $G(x, u)$.

Hence we have shown that $h(x) = (1/(n-1)!)(d^{n-1}/dy^{n-1})(p(y) L(x, y)) |_{y=y_q}$ is in the gradient space of $g(x) = (1/(n-3)!)(d^{n-3}/dy^{n-3})(p(y) L(x, y)) |_{y=y_q}$. The general case is obtained by considering the analog of (1.16). This also covers part (a) of the lemma. ∎

LEMMA 2.5. *Let $f(x) \in C[0, 1]$ and vanish only at a finite number of points with $S^-(f) = k$. Let H represent the $d(g)$-dimensional subspace of all functions of form* (2.8) *satisfying* (2.7). *If $d(g) \geqslant k + 1$ then there is an $h_1 \in H$ such that $\int_0^1 h_1 f < 0$.*

Proof. We assume $d(g) = k + 1$, for if not our analysis would apply to a suitably chosen subspace of H.

We rewrite (2.8) as

$$h(x) = \sum_{i=0}^{s} \sum_{j=0}^{m_i} h_{ij} \frac{(x - y_i)_+^{n-1-j}}{(n-1-j)!} \tag{2.14}$$

with $y_0 = 0$ and $m_0 = n - 1$. Note

$$\sum_{i=0}^{s} (m_i + 1) = M, \qquad m_i \leqslant n - 1.$$

For each $\epsilon > 0$, by smoothing and by Lemma 1.4, we obtain $h_\epsilon(x)$.

$$h_\epsilon(x) = \sum_{i=0}^{s} \sum_{j=0}^{m_i} h_{ij}(\epsilon)\, K_\epsilon^{(j)}(x, y_i) \tag{2.15}$$

such that

$$h_\epsilon f \leqslant 0 \qquad \text{and} \qquad \int_0^1 h_\epsilon f < 0.$$

(If $h_{ij}(\epsilon) = h_{ij}$, we say $h_\epsilon(x)$ in (2.15) is $h(x)$ of (2.14) smoothed.)

Moreover by multiplication by a suitable constant we assume

$$\| h_\epsilon \| = 1.$$

Since the functions (2.14) as well as the functions (2.15) are clearly linearly independent, it follows from (2.3) that there exists a $Q > 0$, such that (for more details see [5, proof of Theorem 1]):

$$\left\| \sum_{i,j} h_{ij}(\epsilon)\, K_\epsilon^{(j)}(x, y_i) \right\| \geqslant Q \sum | h_{ij}(\epsilon) |, \qquad \epsilon > 0.$$

Thus a subsequence of the $h_\epsilon(x)$ as $\epsilon \to 0$ approaches a $h_1(x) \in H$, which because of (2.3) satisfies the conclusions of the lemma. ∎

It now follows from the previous lemmas that

LEMMA 2.6. *For $1 \leqslant p < \infty$, g optimal implies $S^-(F + g) \geqslant d(g)$.*

Definitions 1.2 and 1.3 for the sign changes and generalized sign changes of a function of the form (2.14) carry over; however, see the note after Definition 1.2.

LEMMA 2.7. *If $h_2(x)$ is of the form* (2.14), *then*

(a) $S^-(h_2) \leqslant V^-(h_2) \leqslant M - 1$.

(b) *If $S^-(h_2) = V^-(h_2) = k$, then* sgn $h_2(x_j) = V_{i_j}$, $j = 0,..., k$, *where the x_j are a maximal set for the sign changes of $h_2(x)$, and the V_{i_j} are a maximal number of coordinates for the sign changes in $V(h_2)$.*

Proof. For a fixed $h_2(x)$, and $\epsilon > 0$, let $h_\epsilon(x)$ be $h_2(x)$ smoothed. Hence for sufficiently small ϵ, we have by (2.4) and Lemma 1.6

$$S^-(h_2) \leqslant S^-(h_\epsilon) \leqslant V^-(h_\epsilon) = V^-(h).$$

This establishes (a), and (b) follows similarly. ∎

With $F(x) = \int_0^1 (x-y)_+^{n-1}\,dy$, and $g(x)$ of the form (2.1) with $m_i \leqslant n-1$, Definition 1.4 for $V(F+g)$, as well as the definitions of $V^-(F+g)$, $D(F+g)$, $C(F+g)$ and $E(F+g)$ (of Lemma 1.7), carry over.

LEMMA 2.8. *With the definitions above,*

(a) $S^-(F+g) \leqslant V^-(F+g) \leqslant D(F+g) - C(F+g) - E(F+g)$;

(b) *if* $S^-(F+g) = V^-(F+g) = k$, *then* $\operatorname{sgn}(F+g)(x_j) = V_j$, $j = 0,\ldots,k$, *where the x_j are maximal set of coordinates for the sign changes of $F+g$, and the V_i are a maximal number of coordinates for the sign changes in $V(F+g)$.*

Proof. The proof again follows by smoothing, since now clearly

$$\lim_{\epsilon\to 0}\,(n-1)!\int_0^1 K_\epsilon(x,y)\,dy \to \int_0^1 (x-y)_+^{n-1}\,dy \text{ uniformly.}$$

LEMMA 2.9. *If g of the form* (2.1) *is optimal, then*

(a) $N = \sum_{i=1}^t m_i$,

(b) *all free knots in* (0, 1) *are of odd multiplicity,*

(c) $S^-(F+g) = d(g) = D(F+g) - C(F+g) = n + \sum_{i=1}^t (m_i+1)$.

Proof. The proof is analogous to the proof of Lemma 8. ∎

LEMMA 2.10. *If g of the form* (2.1) *is optimal then*

(a) *all the free knots of g have multiplicity one,*

(b) *at a free knot y_i of multiplicity one* $\operatorname{sgn} a_{i0} = -1$.

Proof. The proof is analogous to the proof of Lemma 1.9; we merely need apply Lemma 2.4(b) instead of Lemma 1.3(c). For fixed n, we have already ruled out the possibility of a knot of multiplicity n, or of an even multiplicity. Thus if n is odd, the highest multiplicity that can occur is $n-2$. Hence Lemma 2.4(b) is applicable. For n even, a knot of multiplicity $n-1$ could conceivably occur; this is ruled out in the following Lemma. Thus Lemma 2.4(b) is always applicable. This will complete our proof.

LEMMA 2.11. *If $g(x)$ is optimal then no knot of $g(x)$ has multiplicity $n-1$.*

Proof. Say $g(x)$ of (2.1) is optimal and has a knot of multiplicity $n-1$ at y_q, $0 < y_q < 1$. We show that this leads to a contradiction. We know that

$$S^-(F+g) = V^-(F+g) = n + \sum_{i=1}^t (m_i+1), \tag{2.16}$$

which first implies, by Lemma 2.8(b) and the definition of $V(F+g)$, that

$$\operatorname{sgn} r_{q,n-2} < 0. \tag{2.17}$$

We now wish to establish over the open interval $(0, y_q)$

$$S^-(F+g)|_{x\in(0,y_q)} = n + \sum_{i=1}^{q-1} (m_i + 1) = V(F+g)|_{x\in(0,y_q)} \tag{2.18}$$

and over the open interval $(y_q, 1)$

$$S^-(F+g)|_{x\in(y_q,1)} = n + \sum_{i=q+1}^{t} (m_i + 1) = V(F+q)|_{x\in(y_q,1)}. \tag{2.19}$$

Say (2.18) did not hold, but rather

$$S^-(F+g)|_{x\in(0,y_q)} < n + \sum_{i=1}^{q-1} (m_i + 1). \tag{2.18$'$}$$

Let

$$g_-(x) = \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{q-1} \sum_{j=0}^{m_i-1} r_{ij}(x-y)_+^{n-1-j}. \tag{2.19$'$}$$

The gradient space of g_- contains $H(g_-)$ of dimension

$$d(g_-) = n + \sum_{i=1}^{q-1} (m_i + 1)$$

(see Lemma 2.4). We can by Lemma 2.5 find a $h_-(x) \in H(g_-)$ such that

$$\int_0^{y_q} \operatorname{sgn}(F+g)|F+g|^{p-1} h_-(x) < 0.$$

Further, since g has a knot of multiplicity $n-1$ at y_q, the space $H(g)$ of all functions of the form (2.8) contains functions with knots of multiplicity n at y_q. Thus if

$$\begin{aligned} h(x) &= h_-(x), \qquad 0 \leqslant x \leqslant y_q, \\ &= 0, \qquad\qquad x > y_q, \end{aligned}$$

$h(x) \in H(g)$. Hence by Lemma 2.2, g would not be optimal. This rules out (2.18)$'$. Analogous reasoning holds for (2.19). Thus (2.16) implies (2.18) and (2.19) must hold.

Since the last component to the right of $V(F+g)|_{x\in(0,y_q)}$ is $\boxed{+1}$, it follows from Lemma 2.8(b) and (2.18) that

$$(F+g)(x) > 0, \qquad \text{for } \epsilon \text{ sufficiently small and } y_q - \epsilon < x < y_q. \tag{2.20}$$

Equations (2.18) and (2.19) plus (2.16) imply $F + g$ does not change sign at $x = y_q$. Thus (2.20) and (2.17) imply

$$(F + g)(y_q) > 0. \tag{2.21}$$

Further, (2.21) and (2.17) imply the procedure introduced by Rice [18, Theorem 10-3, part (b)] may be used to get a better approximation to F than g in the L_p norm $1 \leqslant p < \infty$. This is a contradiction and the result now follows. ▮

We summarize our results in the following theorem.

THEOREM 2. *For* $1 \leqslant p < \infty$ *there exists a best* L_p *approximation* $g(x)$ *to* $F(x) = \int_0^1 (x - y)_+^{n-1}\, dy$. *Further, each best approximation is of the form*

$$g(x) = \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{N} r_{i0}(x - y_i)_+^{n-1}, \tag{2.22}$$

where

$$a_{n-1} < 0, \qquad r_{i0} < 0, \qquad 0 < y_1 < y_2 < \cdots < y_N < 1. \tag{2.23}$$

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